**The field axioms**

**Closure of F under addition and multiplication**

For all a, b in F, both a + b and a · b are in F (or more formally, + and · are binary operations on F).

**Associativity of addition and multiplication**

For all a, b, and c in F, the following equalities hold: a + (b + c) = (a + b) + c and a · (b · c) = (a · b) · c.

**Commutativity of addition and multiplication**

For all a and b in F, the following equalities hold: a + b = b + a and a · b = b · a.

**Existence of additive and multiplicative identity elements**

There exists an element of F, called the additive identity element and denoted by 0, such that for all a in F, a + 0 = a. Likewise, there is an element, called the multiplicative identity element and denoted by 1, such that for all a in F, a · 1 = a. To exclude the trivial ring, the additive identity and the multiplicative identity are required to be distinct.

**Existence of additive inverses and multiplicative inverses**

For every a in F, there exists an element −a in F, such that a + (−a) = 0. Similarly, for any a in F other than 0, there exists an element a−1 in F, such that a · a−1 = 1. (The elements a + (−b) and a · b−1 are also denoted a − b and a/b, respectively.) In other words, subtraction and division operations exist.

**Distributivity of multiplication over addition**

For all a, b and c in F, the following equality holds: a · (b + c) = (a · b) + (a · c).

**Proposition 1.14 from Rudin.** The axioms for addition imply the following statements.

1. If $x+y=x+z$ then $y=z$ (cancellation)
2. If $x+y=x$ then $y=0$ (uniqueness of the additive identity)
3. If $x+y=0$ then $y=-x$ (uniqueness of the additive inverse)
4. $-\left(-x\right)=x$ (double negation)

*Proof*. If $x+y=x+z$, the axioms for addition give

$$y=0+y=\left(-x+x\right)+y=-x+\left(x+y\right)=-x+\left(x+z\right)=\left(-x+x\right)+z=0+z=z.$$

This proves (a). Take $z=0$ in (a) to obtain (b). Take $z=-x$ in (a) to obtain (c). Since $-x+x=0$, (c) (with $–x$ in place of $x$) gives (d).$ ∎$

**Proposition 1.15 from Rudin**. The axioms for multiplication imply the following statements.

1. If $x\ne 0$ and $xy=xz$ then $y=z$ (cancellation)
2. If $x\ne 0$ and $xy=x$ then $y=1$ (uniqueness of the multiplicative identity)
3. If $x\ne 0$ and $xy=1$ then $y=1/x$ (uniqueness of the multiplicative inverse)
4. If $x\ne 0$ then $1/(1/x)=x$.

**Proposition 1.16 from Rudin**. The field axioms imply the following statements, for any $x$, $y$, and $z\in F$.

1. $0x=0$.
2. If $x\ne 0$ and $y\ne 0$ then $xy\ne 0$.
3. $\left(-x\right)y=-\left(xy\right)=x(-y)$.
4. $\left(-x\right)\left(-y\right)=xy$

*Proof*. $0x+0x=\left(0+0\right)x=0x$. Hence 1.14(b) implies that $0x=0$, and (a) holds. Next, assume $x\ne 0$, $y\ne 0$, but $xy=0$. Then (a) gives

$$1=\left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy=\left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0=0$$

a contradiction. Thus (b) holds. The first equality in (c) comes from

$$\left(-x\right)y+xy=\left(-x+x\right)y=0y=0,$$

combined with 1.14(c); the other half of (c) is proved in the same way. Finally,

$$\left(-x\right)\left(-y\right)=-\left[x\left(-y\right)\right]=-\left[-\left(xy\right)\right]=xy$$

by (c) and 1.14(d).$∎$