The Soda Mixing Problem

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Statement of the Problem. Two jars contain an equal volume of soda. One contains Sprite, the other Coca Cola. You take a small amount of Coca Cola from the Coca Cola jar and add it to the Sprite jar. After uniformly mixing this concoction, you take a small amount out and put it back in the Coca Cola jar, restoring both jars to their original volumes. After having done this, is there more Coca Cola in the Sprite jar or more Sprite in the Coca Cola jar? Or, are they equally contaminated?

Solution. Let Jar C be the jar initially containing Coke and Jar S be the jar initially containing Sprite. Let V be the volume of the liquid in each container and let x be the (small) volume to be removed from the jar containing Coke. Before the mixing begins, the volume of each kind of soda in each jar is simply V. (See Table 1.) Keep in mind that Jar C will eventually have both Coke *and* Sprite, and likewise for Jar S.

During the mixing, the volume of Coke in Jar C is $V - x$ because we defined x to be the volume of Coke removed. Likewise, the volume of coke in Jar S is now x.

After the mixing, the volume of Coke in Jar C is the volume of Coke the was present during the mixing *and* the volume of Coke added when x units from Jar S are put back in Jar C. How can we express the amount of Coke added back in? It is clearly some fraction of x.

Since we assume the contents of Jar S were uniformly mixed, the fraction of Coke in Jar S and any sample taken from it will be

$$\text{Concentration of Coke} = \frac{\text{volume of Coke}}{\text{total volume}} = \frac{x}{V+x}$$

Likewise, the fraction of Sprite in Jar S and any sample taken from it will be

$$\text{Concentration of Sprite} = \frac{\text{volume of Sprite}}{\text{total volume}} = \frac{V}{V+x}$$

Thus, the volume of Coke in the sample we take out of Jar S is $x \cdot (\text{Concentration of Coke}) = x \cdot (x/(V+x))$ and the volume of Sprite in the sample we take out of Jar S is $x \cdot (\text{Concentration of Sprite}) =$

Table 1: Volume of soda in each jar

	Jar C		Jar S	
	Coke	Sprite	Coke	Sprite
Before Mixing	V	0	0	V
During Mixing	$V - x$	0	x	V
After Mixing	$(V-x) + x\left(\frac{x}{x+V}\right)$	$0 + x\left(\frac{V}{V+x}\right)$	$x - x\left(\frac{x}{V+x}\right)$	$V - x\left(\frac{V}{V+x}\right)$

$x \cdot (V/(V+x))$. These amounts are then added to the Coke and Sprite volumes in Jar C and subtracted from the Coke and Sprite volumes in Jar S. (See Table 1.)

We now simplify the expressions in the last row of the table. The first expression that gives the volume of Coke in the Jar C, for instance, gives

$$\begin{aligned}(V-x)+x\left(\frac{x}{x+V}\right) &= \frac{(V-x)(V+x)}{V+x}+\frac{x^2}{V+x} \\ &= \frac{V^2-x^2}{V+x}+\frac{x^2}{V+x} \\ &= \frac{V^2}{V+x}\end{aligned}$$

Simplifying all the expressions in the last row of the table in like manner, we can see the clear answer to the question. The question asks us to compare the volume of Sprite in the jar initially containing Coke (Jar C) to the volume of Coke in the jar initially containing Sprite (Jar S). Looking at the table, it is clear they are the same.

	Jar C		Jar S	
	Coke	Sprite	Coke	Sprite
After Mixing	$\frac{V^2}{V+x}$	$\frac{xV}{V+x}$	$\frac{xV}{V+x}$	$\frac{V^2}{V+x}$